Eur. Phys. J. C 9, 633–642 (1999)
Digital Object Identifier (DOI) 10.1007/s100529900090

THE EUROPEAN
PHYSICAL JOURNAL C

Recovering corrections in the analysis of intermittent data

B. Ziaja

Department of Theoretical Physics, Institute of Nuclear Physics, Radzikowskiego 152, PL-31-342 Cracow, Poland
(e-mail: beataz@qcd.ifj.edu.pl)

Received: 25 March 1999 / Published online: 30 June 1999

Abstract. The analysis of intermittent data is improved. It is proven that the standard method of recovering the history of a particle cascade generally does **not** reproduce the structure of the true cascade. The **recovering corrections** to the standard method are proposed and tested in the framework of multiplicative cascading models.

1 Introduction

The first data on possible intermittent behaviour in multiparticle production [1] came from the analysis of the single event of high multiplicity recorded by the JACEE Collaboration [2]. Data from later accelerator experiments [3] confirmed that there are large dynamical fluctuations appearing in the high-energy multiparticle final states that manifest a scaling behaviour. Many different models have been proposed since to explain the effect [4]. Some of them suggest that an underlying final-state multiparticle cascade may be responsible for the scaling of multiparticle moments [4]. In this approach, the intermittent data represent the last stage of the cascade, and the main problem lies in the extraction of the information on previous cascading stages, which is in some way encoded in the last-stage data. It should be stressed that the problem of recovering the history of the cascade may not be solvable if addressed generally. However, the self-similar processes which are assumed to underlie the final-state structure obey some scaling law. This makes many features of local dynamics disappear, and one may expect to extract from the last-stage data at least a part of information on the real cascade parameters.

The method of recovering the history of the cascade from the last-stage data was proposed and applied originally to the JACEE event data. Since that time has become a standard tool of multiparticle data analysis [4], especially in the event-by-event analysis [5].

In this paper, we would like to improve the standard method of analyzing the intermittent data, taking into account corrections due to recovering the history of the particle cascade in the framework of multiplicative random cascading models [6,7]. This problem has already been addressed and analyzed in part in [8]. Our discussion will proceed as follows. In Sect. 2, we characterize briefly the standard method of estimation of intermittency exponents, and introduce the definition of recovering corrections. In Sect. 3, the definition of multiplicative models is presented, and the special cases of multiplicative models, α, p, and $(p+\alpha)$, are summarized. In Sect. 4 the recursive equation for recovering corrections in a multiparticle model with possible neighbour-node memory is derived. Section 5 is devoted to the implementation of recovering corrections into the analysis of data. The implementation algorithm is proposed and numerically tested. Finally, in Sect. 6 we present our conclusions.

2 Standard estimation of intermittency exponents and recovering corrections

Consider a sample of M bins describing an individual (intermittent) event. For simplicity, we assume that $M = 2^n$, where n is a natural number. We thus have 2^n numbers describing the content of each bin:

$$x_i^{(n)}, \;\; i = 0, 1, \ldots, 2^n - 1 \tag{1}$$

which represent, e.g., the distribution of particle density into bins. One assumes that the bin ensemble has been generated in some cascading process, and that the **unnormalized density moments** $z_q^{(n)}$ for this process,

$$z_q^{(n)} = \frac{1}{2^n} \sum_{i=0}^{2^n-1} \left(x_i^{(n)}\right)^q, \tag{2}$$

manifest a scaling behaviour parametrized by intermittency exponents ϕ_q:

$$z_q^{(n)} \sim 2^{n\times\phi_q}. \tag{3}$$

The standard method of estimation of intermittency exponents was introduced first for the analysis of JACEE events [2]. The method recovered the history of the cascade in the following manner: it established the value of density moments for each cascade step, and made the linear χ^2 fit to the points $(k, \log z_q^{(k)})$ $(k = 1, \ldots, n,\ \log x \equiv \log_2 x)$:

$$\log z_q^{(k)} = k \times \phi'_q + b. \tag{4}$$

In this way, the eventual long-range correlations could be separated and would not contribute to the estimated slope ϕ_q. For the assumed bin-into-two-bins splitting scheme, the true value of the $x_i^{(n-k)}$ bin content was replaced by $y_i^{(n-k)}$:

$$y_i^{(n-k)} = \frac{1}{2^k} \sum_{j=0}^{2^k-1} x^{(n)}_{2^k \times i+j}. \tag{5}$$

The intermittency exponents were extracted from the reconstructed moments $z^{(k)}_{q;\,\mathrm{rec.}}$:

$$z^{(k)}_{q;\,\mathrm{rec.}} = \frac{1}{2^k} \sum_{j=0}^{2^k-1} \left(y_j^{(k)}\right)^q, \tag{6}$$

assuming their power-law behaviour:

$$z^{(k)}_{q;\,\mathrm{rec.}} \sim 2^{k \times \phi'_q}. \tag{7}$$

So far (see, e.g., [10]) the value of (normalized) ϕ_q has been estimated simply with the assumption that $\phi_{q;\mathrm{norm.}} = \phi'_q$ $(\phi_{q;\,\mathrm{norm.}} := \phi_q - q\phi_1)$.

There is, however, an open question of how the cascade recovered from data refers to the true cascade which generated the data. For the purpose of estimating the intermittency exponents, it is enough to ask about the relation between the true density moments z_q and the reconstructed ones obtained from (6). It is obvious that formula (5) loses a piece of information on the primary cascade. We give a simple example to illustrate the problem. If we assume that the underlying cascading process preserves, e.g., the total particle density $\sum_{j=0}^{2^k-1} x_j^{(k)} = 1$ $(k = 1, \ldots, n)$, then it follows from (6) that the reconstructed cascade will not manifest this property: $\sum_{j=0}^{2^k-1} y_j^{(k)} \neq 1$. This means that the standard method does not recover the conservation law present in the true cascade.

Moreover, it was found explicitly for the α model [6,7] (which does not preserve the total particle density) that there exists always a discrepancy between the true value of ϕ_q and its estimation ϕ'_q (7), due to the recovering technique (5). This problem is discussed in detail in [11].

We may express the discrepancy between the true and the reconstructed moments at the $(n-k)$th cascade step in a following way:

$$z^{(n-k)}_{q;\,\mathrm{rec.}} = z_q^{(n-k)} \times p_q(k), \tag{8}$$

where the factor $p_q(k)$ denotes the corrections due to recovering procedure (5); we will call them recovering corrections. Corrections $p_q(k)$ contain information on the parameters of a specific process that has generated the true cascade. It is obvious that they depend also on the cascade step. Substituting (8) into (4), one arrives at the relation:

$$\log z^{(n-k)}_{q;\,\mathrm{rec.}} - \log(p_q(k)) = (n-k) \times \phi_{q;\,\mathrm{corr.}} + b, \tag{9}$$

where the fitted slope $\phi_{q;\,\mathrm{corr.}}$ estimates the true intermittency exponent ϕ_q.

In this paper, we confine ourselves to recovering corrections considered for the class of multiplicative random cascading models. For the multiplicative cascade ("multiplicative" means that at each cascade step, the bin content is multiplied by a number to generate the bin content at the next cascade step) relation (8) holds explicitly, and recovering corrections take the form [11] (for proof, see Appendix A):

$$p_q(k) = \left\langle \left(\frac{1}{2^k} \sum_{i=0}^{2^k-1} x_i^{(k)} \right)^q \right\rangle \tag{10}$$

where the average $\langle \ldots \rangle$ is taken over the (eventual) random choices while generating the cascade. The starting bin $x_0^{(0)}$ is set equal to 1. It is worth noticing that $p_q(k)$ may be also expressed in terms of the erraticity moments [9]:

$$p_q(k) = C_{1\,,\,q}. \tag{11}$$

3 Multiplicative models

As already mentioned, in this paper we restrict ourselves to the class of multiplicative random cascading processes with possible neighbour-node memory, and generate the uniform distribution of particle density. The commonly used models of random cascading, the α model [12] and p model [13], belong to this class. For the purpose of testing our predictions for recovering corrections, we introduce below a new, many-parameter $(p + \alpha)$ model.

In the multiplicative random cascading processes with possible neighbour-node memory, we assume for simplicity the root of a cascade to be equal to 1: $x_0^{(0)} = 1$. One generates the next stages of the cascade recursively. The scheme is the following. The two bins $x_{2i}^{(k+1)}$ and $x_{2i+1}^{(k+1)}$ are obtained from $x_i^{(k)}$ by multiplication,

$$\begin{aligned} x_{2i}^{(k+1)} &:= W_1 \times x_i^{(k)}, \\ x_{2i+1}^{(k+1)} &:= W_2 \times x_i^{(k)}, \end{aligned} \tag{12}$$

where W_1 and W_2 are random variables of the m model parameters a_j, $j = 1, \ldots, m$:

$$\begin{aligned} W_1 &= a_j \text{ with probability } p_{a_j}, \\ W_2 &= a_j \text{ with probability } p_{a_j}, \end{aligned} \tag{13}$$

with normalized probability weights p_{a_j}, where

$$\sum_{j=1}^{m} p_{a_j} = 1. \tag{14}$$

The distribution of particle density will be uniform if the following condition is fulfilled:

$$p(W_1 = a_i, W_2 = a_j) = p(W_1 = a_j, W_2 = a_i), \tag{15}$$

where $p(W_1 = a_i, W_2 = a_j)$ denotes the probability of choosing in (12) $W_1 = a_i$ and $W_2 = a_j$ $(i, j = 1, \ldots, m)$. Then the density moments fulfill relation (3), where intermittency exponents ϕ_q are equal to:

$$\phi_q = \log(a_1^q p_{a_1} + \ldots + a_m^q p_{a_m}). \quad (16)$$

The models α, p, and $(p+\alpha)$ may be derived from the general multiplicative rule (12). To obtain the α model, it is enough to assume random variables W_1, W_2 to be independent:

$$\langle W_1 W_2 \rangle = \langle W_1 \rangle \langle W_2 \rangle. \quad (17)$$

The α model has no node memory; therefore, no conservation law can be implemented here.

Relation (12) reduces to the p model after setting $m = 2$ and:

$$\begin{aligned} a_2 &= 1 - a_1, \\ p_{a_1} &= p_{a_2} = 0.5, \\ p(W_2 = a_2 \mid W_1 = a_1) &= 1, \\ p(W_2 = a_1 \mid W_1 = a_2) &= 1, \end{aligned} \quad (18)$$

where $p(W_2 = a_i \mid W_1 = a_j)$ denotes the conditional probability of $W_2 = a_i$, if $W_1 = a_j$. The p model is an example of a multiplicative model with the neighbour-node memory. Relation (18) implies that the sum $x_{2i}^{(k+1)} + x_{2i+1}^{(k+1)} = x_i^{(k)}$, and the particle density in a node is preserved.

Finally, we introduce the many-parameter $(p+\alpha)$ model, using relation (12) combined with the α- and p-model restrictions:

$$\begin{aligned} a_{2i} &= 1 - a_{2i-1}, \\ p_{a_{2i}} &= p_{a_{2i-1}}, \end{aligned} \quad (19)$$

where m is an even number $(i = 1, \ldots, (m/2))$, and:

$$\begin{aligned} p(W_2 = a_{2i} \mid W_1 = a_{2i-1}) &= 1, \\ p(W_2 = a_{2i-1} \mid W_1 = a_{2i}) &= 1. \end{aligned} \quad (20)$$

One may check that the particle distribution generated in the $(p+\alpha)$ model is uniform. The $(p+\alpha)$ model may involve any number of parameters a_i. Therefore it describes a more realistic case of cascading, since for large m, the distributions of particle density for W_1 and W_2 (13) may be approximated by a continuous distribution function $f(x)$: $W_{1,2} = x$ with probability $f(x)\,\mathrm{d}x$. The total particle density will be preserved for any m, according to (20).

4 Recovering corrections in multiplicative models

Now we calculate explicitly the correction $p_q(k)$ for any multiplicative model with possible neighbour-node memory. To do this, we will split the bins (x_i's) appearing in (10) into a left half $(i < 2^{k-1})$ and a right half $(i \geq 2^{k-1})$ [11]:

$$\begin{aligned} p_q(k) &= \left\langle \left(\frac{1}{2^k} \sum_i l_i + r_i \right)^q \right\rangle \\ &= \frac{1}{2^q} \left\langle \sum_{j=0}^{q} \binom{q}{j} \left(\frac{1}{2^{k-1}} \sum_i l_i \right)^j \right. \\ &\quad \times \left. \left(\frac{1}{2^{k-1}} \sum_i r_i \right)^{q-j} W_1^j W_2^{q-j} \right\rangle. \end{aligned} \quad (21)$$

Using the fact that the left and right bins are independent, one arrives at the recurrence equation

$$p_q(k) = \frac{1}{2^q} \sum_{j=0}^{q} \binom{q}{j} p_j(k-1) p_{q-j}(k-1) \langle W_1^j W_2^{q-j} \rangle \quad (22)$$

which may be solved recursively together with the initial data:

$$\begin{aligned} p_q(0) &= 1, \\ p_0(k) &= 1. \end{aligned} \quad (23)$$

A similar recurrence relation has also been obtained in [8]. It should be stressed that coefficients $\langle W_1^j W_2^{q-j} \rangle$ are the only parameters of the model needed to solve (22) recursively. This means that to calculate $p_q(k)$ for a given model we need only to know the coefficients $\langle W_1^j W_2^{q-j} \rangle$. In the next section, we show how to apply this observation to the data analysis.

Finally, we present recovering corrections calculated for the α and p models:

$$\begin{aligned} p_q^{\alpha\text{-}model}(k) &= \frac{1}{2^q} \sum_{j=0}^{q} \binom{q}{j} p_j(k-1) p_{q-j}(k-1) \\ &\quad \times 2^{\phi_j + \phi_{q-j}}, \quad (24) \\ p_q^{p\text{-}model}(k) &= 2^{\phi_1 qk} \quad (25) \end{aligned}$$

where ϕ_j denote intermittency exponents (16).

5 Implementation of the recovering corrections

The idea for how to implement the corrections $p_q(k)$ into the analysis of the α-model data was sketched briefly in [11]. Here we extend the primary scheme and apply it to the multiplicative model data. As was mentioned in the previous section, coefficients $\langle W_1^j W_2^{q-j} \rangle$ are the only parameters of the model needed to calculate recursively the corrections $p_q(k)$. Let us introduce a new notation for $\langle W_1^j W_2^l \rangle$:

$$\langle W_1^j W_2^l \rangle \equiv k_{j,l}. \quad (26)$$

We ask now how to derive $k_{j,q-j}$ from the model. One may notice that for either $j=0$ or $l=0$, coefficients $k_{j,l}$ equal:

$$k_{j,0} = k_{0,j} = 2^{\phi_j}, \tag{27}$$

where ϕ_j are ordinary intermittency exponents (3) which may be determined from relations (4), (9). To find the value of $k_{j,l}$ $(j,l \neq 0)$ we use the **unnormalized density correlators** $c_{j,l}^{(k)}$ [1,4,14]:

$$c_{j,l}^{(k)} = \frac{1}{2^{k-1}} \sum_{i=0}^{2^{k-1}-1} \left(x_{2i}^{(k)}\right)^j \left(x_{2i+1}^{(k)}\right)^l . \tag{28}$$

In multiplicative models, the correlators and the density moments fulfill the relation

$$c_{j,l}^{(k)} = z_{j+l}^{(k-1)} \times k_{j,l}, \tag{29}$$

which can also be rewritten as:

$$\log c_{j,l}^{(k)} = (k-1)\phi_{j+l} + \log k_{j,l}. \tag{30}$$

Relation (29) may be easily derived from (12); since each term in (28) originates from one node $x_i^{(k-1)}$, it can be rewritten as: $\left(x_{2i}^{(k)}\right)^j \left(x_{2i+1}^{(k)}\right)^l = \left(x_i^{(k-1)}\right)^{j+l} W_1^j W_2^l$. Relations (29) and (30) imply that we may derive $k_{j,l}$ in a straigthforward way by calculating correlators and density moments from data, and applying to them the standard χ^2 fit.

Applying the standard method to the correlators at the previous cascade stages, we expect to find a discrepancy (due to the recovering procedure) between reconstructed correlators and the true ones, similarly as was found for the density moments. It can be proven (see Appendix B) that this discrepancy may be expressed in terms of recovering correction $p_q(k)$ (10):

$$c_{j,l;\,\mathrm{rec.}}^{(n-k)} = c_{j,l}^{(n-k)} p_{j+l}(k). \tag{31}$$

Now we have all the tools needed for implementation of recovering corrections in the multiplicative data analysis. Below, we propose an implementation algorithm which recursively adjusts the primary parameters ϕ_q, $k_{j,l}$ $(j+l=q,\ jl>0)$ obtained after applying the standard method to the data:

(**INPUT**) parameters $\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_{q-1}, \boldsymbol{k}_{j,l}$ $(j+l = 1, \ldots, q-1)$, obtained after applying the implementation algorithm for $q = 1, 2, \ldots, q-1$ step by step (for determination of ϕ_1, see Appendix C.iii).

(**1**) Derive $\boldsymbol{\phi}'_q$, $\boldsymbol{k}'_{j,q-j}$ $(j = 1, \ldots, q-1)$ from data, using the standard method, i.e., reconstruct the cascade using (5) and derive the parameters from relations:

$$\log z_{q;\,\mathrm{rec.}}^{(k)} = k \times \phi'_q + b. \tag{32}$$

$$c_{j,l;\,\mathrm{rec.}}^{(k)} = z_{j+l;\,\mathrm{rec.}}^{(k-1)} \times k'_{j,l} \tag{33}$$

where $k = 1, \ldots, n$ (cf. (4), (29)).

(**2**) Derive $\boldsymbol{\phi}_{q;\,\mathrm{corr.}}$, $\boldsymbol{k}_{j,q-j;\,\mathrm{corr.}}$ $(j = 1, \ldots, q-1)$ in the following substeps:

(**2.0**) calculate $p_q(k)$ from the relation (cf. (22))

$$p_q(k) = \frac{1}{2^q} \sum_{j=0}^{q} \binom{q}{j} p_j(k-1) p_{q-j}(k-1) k_{j,q-j}; \tag{34}$$

using $\boldsymbol{\phi}'_q$, $\boldsymbol{k}'_{j,q-j}$, derived in step (**1**), and estimate $\boldsymbol{\phi}_{q;\,\mathrm{corr.}}$ from (cf. (9)):

$$\log z_{q;\,\mathrm{rec.}}^{(n-k)} - \log(p_q(k)) = (n-k) \times \phi_{q;\,\mathrm{corr.}} + b; \tag{35}$$

(**2.1**) Calculate $p_q(k)$ from (34) using $\phi_{q;\,\mathrm{corr.}}$ (other parameters as after step (**1**)), and estimate $\boldsymbol{k}_{1,q-1;\,\mathrm{corr.}}$ from relation (cf. (30),(31), see also Appendix C):

$$\log c_{j,l;\,\mathrm{rec.}}^{(n-k)} - \log(p_{j+l}(k)) = (n-k-1)\phi_{j+l} + \log k_{j,l;\,\mathrm{corr.}}, \ldots, \tag{36}$$

(**2.q-1**) Calculate $p_q(k)$ from (34), using all previously derived parameters $\boldsymbol{\phi}_{q;\,\mathrm{corr.}}$, $\boldsymbol{k}_{j,q-j;\,\mathrm{corr.}}$, and estimate $\boldsymbol{k}_{q-1,1;\,\mathrm{corr.}}$ from (36).

(**3**) Compare the values of $\boldsymbol{\phi}'_q$, $\boldsymbol{k}'_{j,q-j}$ and $\boldsymbol{\phi}_{q;\,\mathrm{corr.}}$, $\boldsymbol{k}_{j,q-j;\,\mathrm{corr.}}$ $(j = 1, \ldots, q-1)$. If the relative difference is large, assume:

$$\phi'_q := \phi_{q;\,\mathrm{corr.}},$$
$$k'_{j,q-j} := k_{j,q-j;\,\mathrm{corr.}},$$

and repeat steps **(2)** and **(3)** recursively until the relative difference between parameters before and after step **(2)** is small enough. Then go to the output, assuming $\boldsymbol{\phi}_q := \boldsymbol{\phi}'_q$, $\boldsymbol{k}_{j,q-j} := \boldsymbol{k}'_{j,q-j}$.

(**OUTPUT**) Parameters $\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_q; \boldsymbol{k}_{j,l}$ $(j+l = 1, \ldots, q)$.

Other technical details and problems that may appear when applying the algorithm to data are listed in Appendix C.

We have performed numerical simulations of the α, p and $(p+\alpha)$ models in order to test how the implementation algorithm works in practice. We generated 10000 cascades of the 10-step length for the α and $(p+\alpha)$ models, and one cascade of the 10-step length for the p model[1] for two different parameter sets separately.

The implementation algorithm analyzed the data of the last cascade step. For each event it estimated the value of normalized intermittency exponents $\phi_{2;\,\mathrm{norm.}}$, $\phi_{3;\,\mathrm{norm.}}$

[1] It can be proven that for a given parameter set, the p model generates always the same values of the correlators and density moments.

Table 1. Estimation of normalized intermittency exponents $\phi_{2;\,\mathrm{norm.}}$ and $\phi_{3;\,\mathrm{norm.}}$ and their dispersions for the α model, using the standard method (second column), the improved method with the implementation algorithm (third column), and dedicated α corrections (24) (fourth column), as compared with the theoretical values (first column), performed for two sets of α-model parameters (see Figs. 1, 2)

	theor.	standard	algorithm	$\alpha -$ corr.
a) $\phi_{2;\,\mathrm{norm.}}$	0.0285	0.0251 ± 0.004	0.0246 ± 0.0033	0.0288 ± 0.004
$\phi_{3;\,\mathrm{norm.}}$	0.0813	0.0757 ± 0.010	0.0727 ± 0.009	0.0798 ± 0.0111
b) $\phi_{2;\,\mathrm{norm.}}$	0.322	0.264 ± 0.044	0.253 ± 0.050	0.276 ± 0.051
$\phi_{3;\,\mathrm{norm.}}$	0.807	0.653 ± 0.105	0.666 ± 0.125	0.750 ± 0.131

Table 2. Estimation of normalized intermittency exponents $\phi_{2;\,\mathrm{norm.}}$ and $\phi_{3;\,\mathrm{norm.}}$ and their dispersions for the $(p+\alpha)$ model, using the standard method (second column) and the improved method with the implementation algorithm (third column), as compared with the theoretical values (first column), performed for two sets of $(p+\alpha)$-model parameters (see Figs. 3, 4)

	theor.	standard	algorithm
a) $\phi_{2;\,\mathrm{norm.}}$	0.333	0.322 ± 0.044	0.305 ± 0.066
$\phi_{3;\,\mathrm{norm.}}$	0.832	0.736 ± 0.118	0.720 ± 0.174
b) $\phi_{2;\,\mathrm{norm.}}$	0.177	0.170 ± 0.023	0.173 ± 0.029
$\phi_{3;\,\mathrm{norm.}}$	0.478	0.438 ± 0.069	0.470 ± 0.092

($\phi_{i;\,\mathrm{norm.}} := \phi_i - \mathrm{i} \times \phi_1$), using the standard method (step **(1)**) with recovering corrections included (steps **(2)** and **(3)**. The results are presented in Figs. 1, 2, 3, 4 (for the α and $(p+\alpha)$ models) and in Tables 1, 2.

For the α model, the histograms of $\phi_{2;\,\mathrm{norm.}}$, $\phi_{3;\,\mathrm{norm.}}$ obtained in the standard method and the histograms with recovering corrections included are almost identical. In this case, the recovering corrections can be implemented better when one applies directly the dedicated α-model recovering correction $p_q^{\alpha\text{-}model}(k)$ (24) (see Figs. 1, 2 and Table 1). The different accuracy of both approaches is due to the fact that the random variables W_1, W_2 are independent in the αmodel, and the coefficients $k_{j,q-j}$ (26) are approximated better by the product $2^{\phi_j} \times 2^{\phi_{q-j}}$ than from correlators (29).

However, the implementation algorithm works well for the $(p+\alpha)$ model (see Figs. 3, 4 and Table 2). For the $(p+\alpha)$ model, the histogram with the recovering corrections included approximates well the theoretical value of normalized intermittency exponent. The histogram obtained by using the standard method is moved slightly to the left in comparison to the histogram with recovering corrections included.

We have checked that for the p model, the theoretical values of normalized intermittency exponents are estimated perfectly by both the standard method and and implementation algorithm, as we have expected[2].

[2] It follows from relations (8), (25) that in the p model, $\phi_{i;\,\mathrm{rec.}\,,\mathrm{norm.}} = \phi_{i;\,\mathrm{norm.}}$ if $i > 1$.

It should also be mentioned that the histograms generated by both the implementation algorithm and dedicated recovering corrections are symmetric, in contrast to the standard ones, and their dispersions are relatively small (see Tables 1, 2). Finally, we notice that the accuracy of the estimation of intermittency exponents, while applying the standard method and the improved one to a given model, depends on the parameters of this model (cf. Figs. 1, 2 and Figs. 3, 4). In Appendix D, we present a qualitative analysis of the effect for intermittency exponents of the second rank. A similar analysis has also been done in [8].

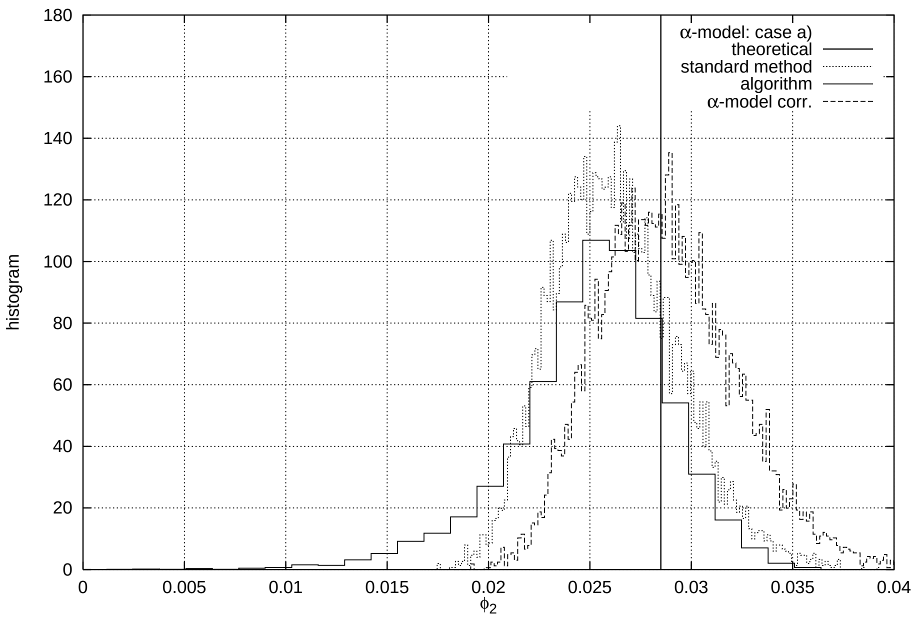
6 Conclusions

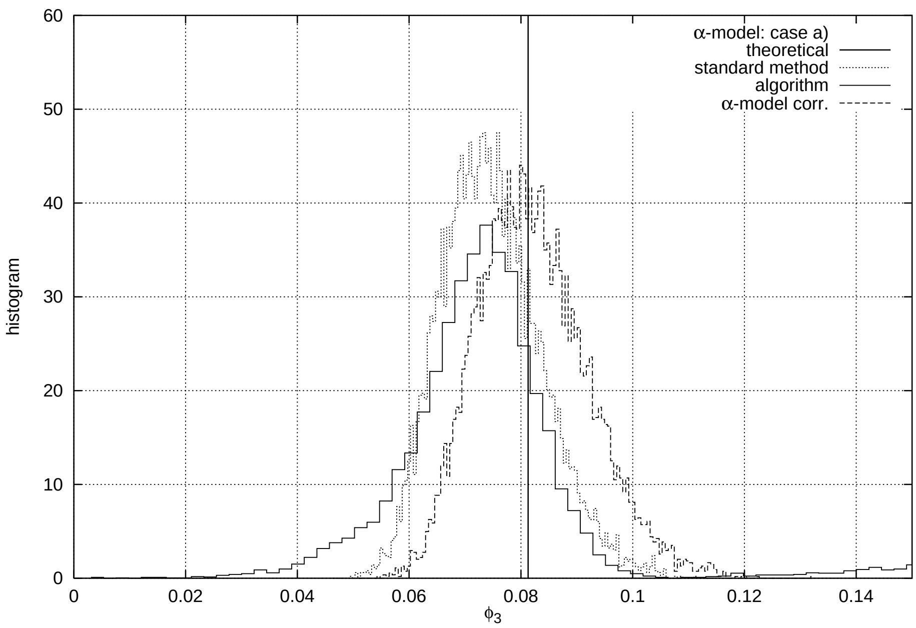
To sum up, we have analyzed the estimation of intermittency exponents from the data that were generated by a multiplicative random cascading process. The following methods were applied: the standard method of cascade recovering (5), and the improved method, which recursively included the recovering corrections. The improved method was applied in the form of the implementation algorithm. Numerical simulations have been performed to check how both methods work in practice. The conclusions may be summarized as follows:

(a) The standard method of estimation of intermittency exponents does not apply for the whole class of multiplicative models: its accuracy depends on the specific properties of the model and its parameters. The method does not detect a conservation law if present in the model.

(b) The improved method of estimation of intermittency exponents applied in the form of recursive implementation algorithm either corrects the standard method estimation or does not change the standard method result. In the latter case, the estimation may be corrected by applying the dedicated recovering corrections. In any case, the improved method tests the applicability of the standard method, and allows one to estimate the accuracy of the intermittency exponent estimation.

While applying the improved method, the parameters of the model are adjusted recursively from the primary (standard method) parameters. The histograms generated by the improved method are symmetric, and their disper-





Figs. 1, 2. Estimation of normalized intermittency exponents $\phi_{2;\,\mathrm{norm.}}$ and $\phi_{3;\,\mathrm{norm.}}$ for the α model, using the standard method (dotted line), the improved method with the implementation algorithm (thin solid line), and dedicated α corrections (24) (dashed line), as compared with the theoretical values (solid line), performed for two sets of α-model parameters:
(a) $a_1 = 0.8$, $a_2 = 1.1$, $p_1 = 1/3$
(b) $a_1 = 0.5$, $a_2 = 1.5$, $p_1 = 1/2$

sions are of the same order as those determined for the standard method. The improved method takes into account the neighbour-node memory (a conservation law), if present in the model, by including the density correlators into the estimation scheme.

Acknowledgements. I would like to thank Prof. A. Białas many suggestions and comments on the manuscript and Dr. R. Janik for discussions. This work was supported in part by Polish Government Grant Project (KBN) 2P03B04214.

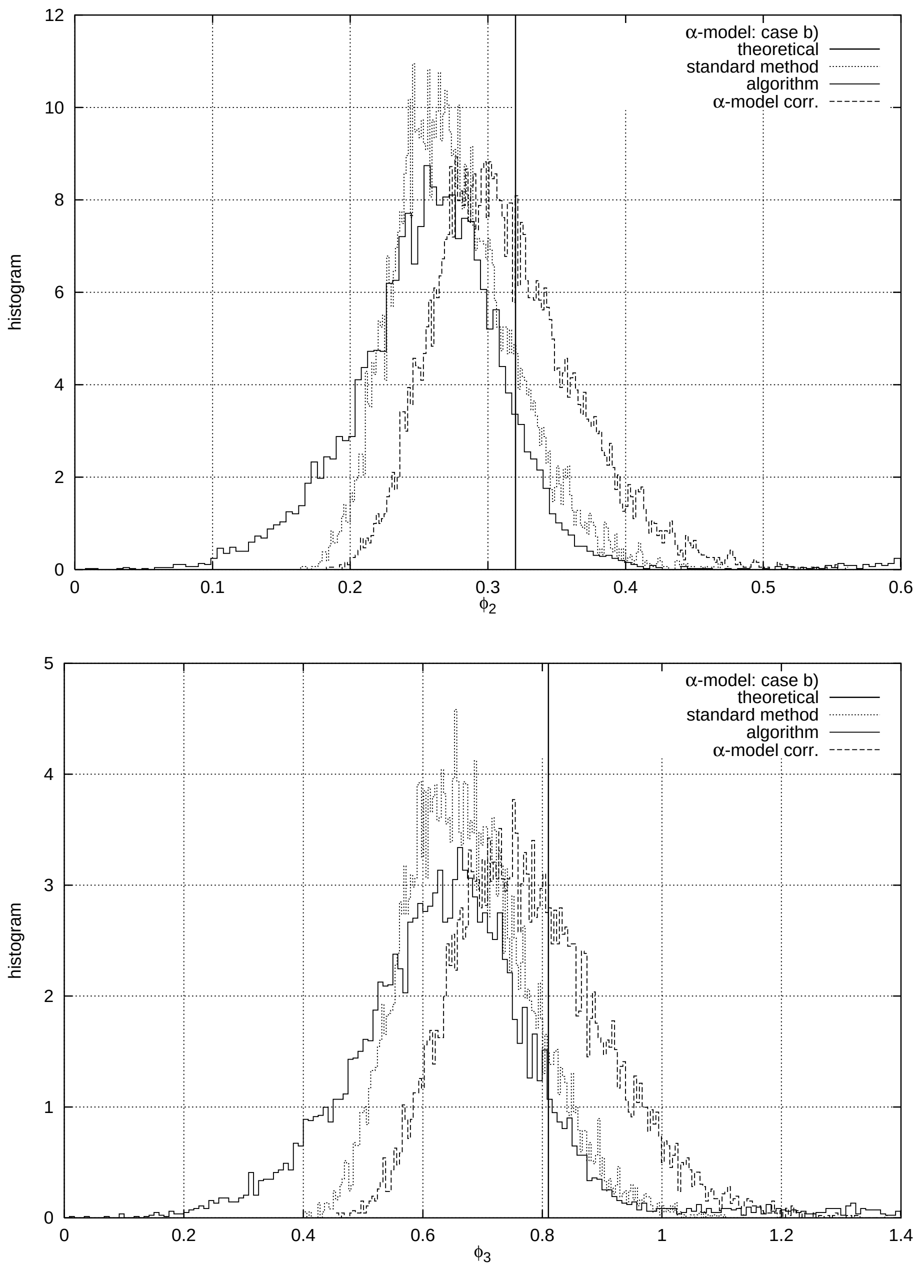
Appendix A

We prove relation (10). The density moment $z_{q;\,\mathrm{rec.}}^{(n-k)}$ may be rewritten as:

$$z_{q;\,\mathrm{rec.}}^{(n-k)} = \frac{1}{2^{n-k}} \sum_{i=0}^{2^{n-k}-1} \left(y_i^{(n-k)} \right)^q = \frac{1}{2^{n-k}} \sum_{i=0}^{2^{n-k}-1} \left(\frac{1}{2^k} \sum_{j=0}^{2^k-1} x_{2^k \times i+j}^{(n)} \right)^q . \quad (37)$$

Notice that:

$$x_{2^k \times i+j}^{(n)} = x_i^{(n-k)} \times x_j^{(k)} . \quad (38)$$



Figs. 1, 2. (continued)

Substituting (38) into (37), one arrives at

$$z^{(n-k)}_{q;\,\mathrm{rec.}} = \frac{1}{2^{n-k}} \sum_{i=0}^{2^{n-k}-1} \left(x_i^{(n-k)}\right)^q \left(\frac{1}{2^k}\sum_{j=0}^{2^k-1} x_j^{(k)}\right)^q = z_q^{(n-k)}\, p_q(k). \quad (39)$$

Appendix B

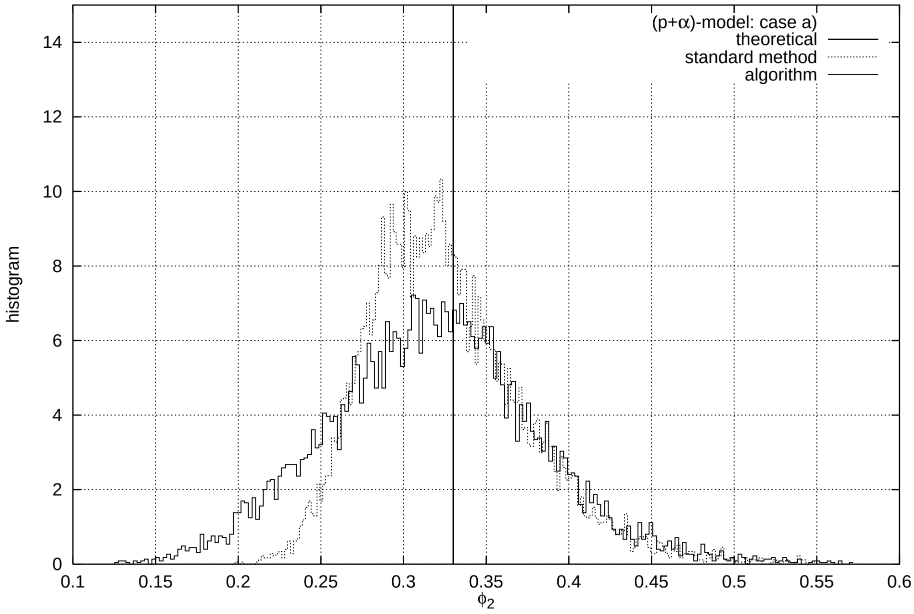
We prove relation (31). The correlator $c^{(n-k)}_{j,l;\,\mathrm{rec.}}$ may be rewritten as:

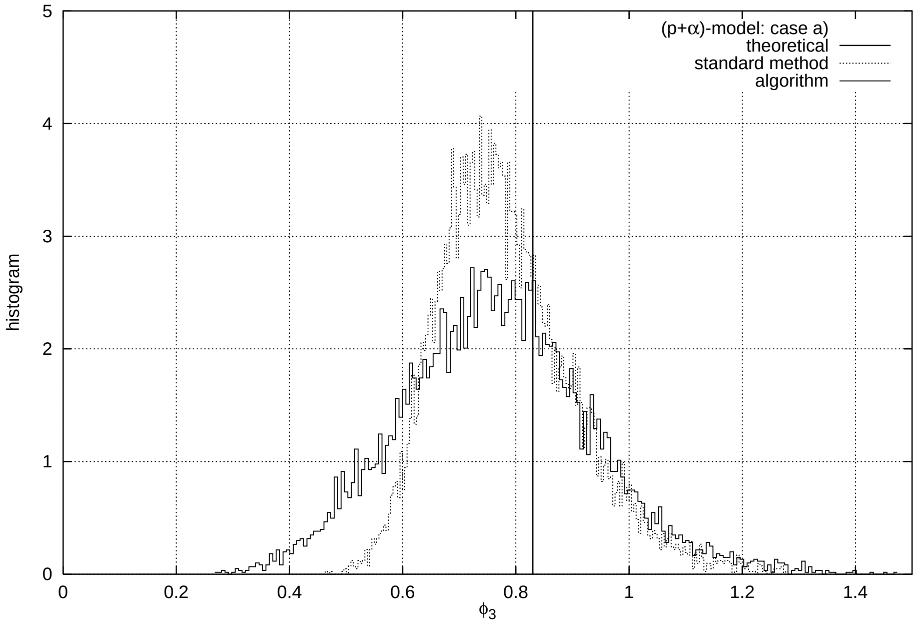
$$c^{(n-k)}_{j,l;\,\mathrm{rec.}} = \frac{1}{2^{n-k-1}} \sum_{i=0}^{2^{n-k-1}-1} \left(y_{2i}^{(n-k)}\right)^j \left(y_{2i+1}^{(n-k)}\right)^l$$

$$= \frac{1}{2^{n-k-1}} \sum_{i=0}^{2^{n-k-1}-1} \left(\frac{1}{2^k}\sum_{m=0}^{2^k-1} x^{(n)}_{2^k\times 2i+m}\right)^j \times \left(\frac{1}{2^k}\sum_{r=0}^{2^k-1} x^{(n)}_{2^k\times(2i+1)+r}\right)^l. \quad (40)$$

Relation (38) implies:

$$c^{(n-k)}_{q;\,\mathrm{rec.}} = \frac{1}{2^{n-k-1}} \sum_{i=0}^{2^{n-k-1}-1} \left(x_{2i}^{(n-k)}\right)^j \left(x_{2i+1}^{(n-k)}\right)^l \times \left(\frac{1}{2^k}\sum_{j=0}^{2^k-1} x_j^{(k)}\right)^{j+l} = c^{(n-k)}_{j,l}\, p_{j+l}(k). \quad (41)$$





Figs. 3, 4. Estimation of normalized intermittency exponents $\phi_{2;\,\mathrm{norm.}}$ and $\phi_{3;\,\mathrm{norm.}}$ for the $(p+\alpha)$ model, using the standard method (dotted line) and the improved method with the implementation algorithm (thin solid line), as compared with the theoretical values (solid line), performed for two sets of $(p+\alpha)$-model parameters:
(a) $a_{2i} = 1 - a_{2i-1}$, $p_{2i} = p_{2i-1} = 0.05$ for $i = 1, \ldots, 10$,
$a_1 = 0.2$, $a_3 = 0.5$, $a_5 = 0.6$, $a_7 = 0.3$, $a_9 = 0.45$,
$a_{11} = 0.25$, $a_{13} = 0.1$, $a_{15} = 0.15$, $a_{17} = 0.87$, $a_{19} = 0.66$;
b) $a_{2i} = 1 - a_{2i-1}$, $p_{2i} = p_{2i-1}$ for $i = 1, \ldots, 10$,
$a_1 = 0.2$, $a_3 = 0.5$, $a_5 = 0.6$, $a_7 = 0.3$, $a_9 = 0.45$,
$a_{11} = 0.25$, $a_{13} = 0.1$, $a_{15} = 0.15$, $a_{17} = 0.87$, $a_{19} = 0.66$,
$2p_1 = 0.05$, $2p_3 = 0.15$, $2p_5 = 0.25$, $2p_7 = 0.40$, $2p_9 = 0.05$,
$2p_{11} = 0.05$, $2p_{13} = 0.02$, $2p_{15} = 0.02$, $2p_{17} = 0.005$, $2p_{19} = 0.005$

Appendix C

We list some technical details which can be useful when applying the implementation algorithm to data, and discuss possible problems.

(i) The recovering corrections applied for calculating ϕ_2, ϕ_3 read:

$$p_1(k) = 2^{k\phi_1} \tag{42}$$

$$p_2(k) = \frac{1}{4}\left(p_2(k-1)k_{2,0} + p_2(k-1)k_{0,2} + 2p_1^2(k-1)k_{1,1} \right) \tag{43}$$

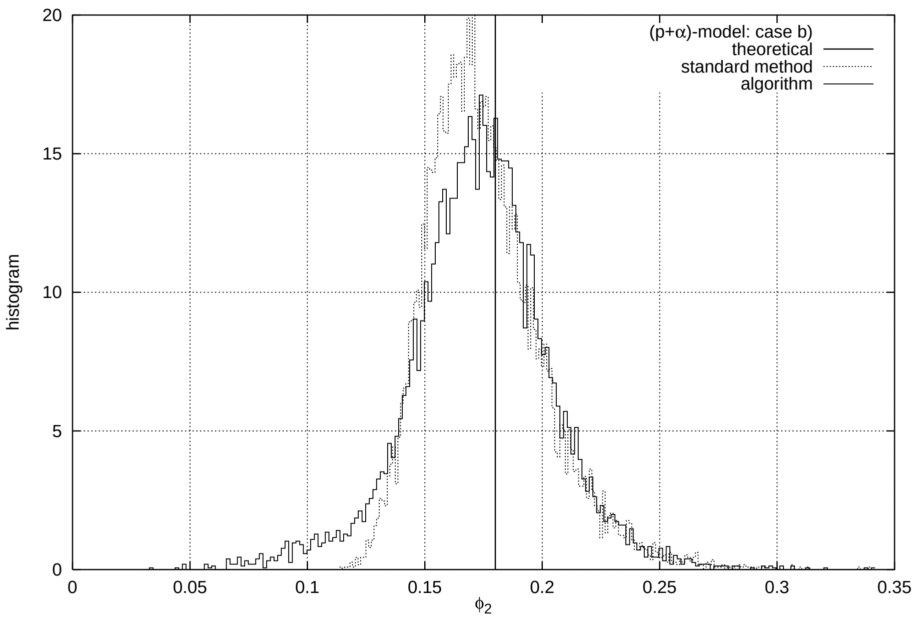
$$p_3(k) = \frac{1}{8}\left(p_3(k-1)k_{3,0} + p_3(k-1)k_{0,3} + 3p_1(k-1)p_2(k-1)k_{1,2} + 3p_2(k-1)p_1(k-1)k_{2,1} \right) \tag{44}$$

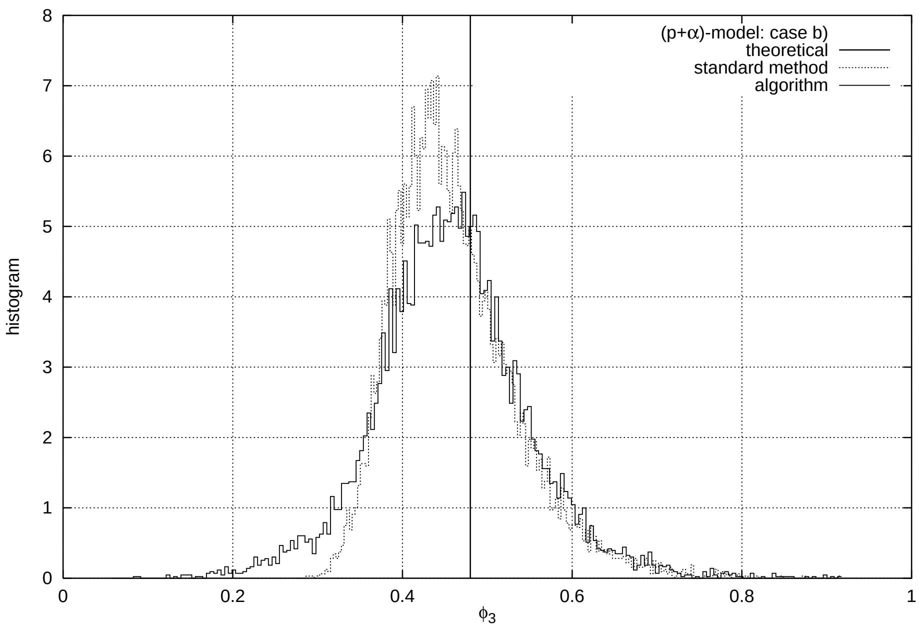
(ii) Since random variables W_1, W_2 generate the uniform distribution (cf. (15)) relations (42)–(44) may be simplified by substituting:

$$k_{j,l} = k_{l,j}. \tag{45}$$

The (experimental) estimation of $k_{j,l}$ will be better if we determine $c_{j,l}^{(n)}$ as:

$$c_{j,l}^{(n)} = \frac{1}{2^{n-1}} \sum_{i=0}^{2^{n-1}-1} \left\{ \left(x_{2i}^{(n)}\right)^j \left(x_{2i+1}^{(n)}\right)^l \right.$$





Figs. 3, 4. (continued)

$$+\left(x_{2i}^{(n)}\right)^{l}\left(x_{2i+1}^{(n)}\right)^{j}\Bigg\}. \quad (46)$$

(iii) On the determination of ϕ_1: It follows from relations (8) and (42) that the unnormalized reconstructed density moment $z_{1;\,\mathrm{rec.}}$ for any multiplicative model takes the form:

$$z_{1;\,\mathrm{rec.}}^{(k)} = 2^{n\phi_1} = z_1^{(n)}, \quad (47)$$

which does not depend on the cascade stage k. Therefore the proposed estimation of ϕ_1, implied by (47) reads:

$$\phi_1 = \frac{\log z_{1;\,\mathrm{rec.}}^{(n)}}{n}. \quad (48)$$

(iv) To estimate coefficients $k_{j,l}$ in step **(1)** of the implementation algorithm, we calculate the reconstructed correlators and density moments, and apply relation (33), whereas in step **(2)**, we use relation (36) to do the same.

The brief explanation of the method is following. In formula (33), $k_{j,l}$ appears as a slope, and an ordinary linear χ^2 fit may estimate it with a good accuracy. This approach works well for the reconstructed correlators and moments.

On the other hand, the long-range correlations $\log b$, present in relation (36), add to the value of $\log k_{j,l}$, and generate a large error while estimating $k_{j,l} := \exp(\log k_{j,l} + \log b)$ from (36). We could try to estimate b, assuming that:

$$z_q^{(n)} = 2^{n\times\phi_q} b. \quad (49)$$

Then it follows from (36) that:

$$\log c_{j,l}^{(n)} = (n-1)\phi_{j+l} + \log \bar{k}_{j,l}, \quad (50)$$

and $\log k_{j,l}$ equals:

$$\log k_{j,l} = \log \bar{k}_{j,l} - \log b. \quad (51)$$

The latter approach does not work for the reconstructed moments, where relations (49) and (50) apply only approximately. However, it applies quite well for the moments with recovering corrections included, because for this case, relation (33) would require including recovering corrections to both correlators and density moments, which in turn would generate a larger error in estimating $k_{j,l}$.

It was checked that the above method works for the multiplicative-model data. However, the problem of the determination of coefficients $k_{j,l}$ and, in particular, the problem of the determination of correlators $c_{j,l}^{(n)}$ from the real data is much more complicated (see, e.g., [4,14]). In this case, the method needs some improvement which we will not discuss here.

(v) The recursive implementation algorithm is not always convergent. Since estimation at the **q**th step depends upon parameters which were adjusted in the previous steps $\mathbf{1}, \ldots, \mathbf{q-1}$, the estimation errors propagate and get larger with growing q. Then it happens sometimes (but not very often) that recursive adjusting ends with the repeating of a sequence of different values of parameters, or parameters become indefinite. In such a case, we stop the algorithm, assuming for the values of intermittency parameters those derived in step **(1)**.

Appendix D

The accuracy of the estimation of intermittency exponents, while applying the standard method and the improved one to a given model, depends on the parameters of this model (cf. Figs. 1, 2 and Figs. 3, 4). Below we present a qualitative analysis of the effect for the intermittency exponents of the second rank.

Equation (43) may be solved analytically, and the solution (valid for any multiplicative model) takes the closed form

$$p_2(k) = (1-A) \times 2^{(\phi_2-1)k} + A \times 2^{2k\phi_1}, \quad (52)$$

where

$$A = \frac{k_{1,1}}{2^{2\phi_1+1} - 2^{\phi_2}}. \quad (53)$$

The reconstructed moments then read:

$$z_{2;\,\mathrm{rec.}}^{(k)} = (1-A)\, 2^{(\phi_2-1)n} \times 2^k + A\, 2^{\phi_1 n} \times 2^{\phi_{2;\,\mathrm{norm.}}\, k}. \quad (54)$$

There are **two** power-law terms: 2^k and $2^{\phi_{2;\,\mathrm{norm.}}\, k}$ in $z_{2;\,\mathrm{rec.}}^{(k)}$. In order to establish how they influence the determination of ϕ_2' (4), (32) we performed the following check. For a given multiplicative model with fixed parameters (e.g., α model with parameters as in Figs. 1, 2) we established the values of $z_{2;\,\mathrm{rec.}}^{(k)}$ from (54), and made the linear χ^2 fit to the points $(k, \log z_{2;\,\mathrm{rec.}}^{(k)})$. The slope obtained from the fit estimated the value of ϕ_2'. For the α model, we have obtained, for case (a), $\phi_2' = 0.0250$, and for case (b), $\phi_2' = 0.284$. Both results agree with the ϕ_2' obtained from the model simulation (cf. Figs. 1,2 and Table 1). A similar analysis can be also peformed for the $(p+\alpha)$ model.

The above results confirm our observation that the accuracy of the estimation of intermittency exponents from the standard method depends on the parameters of the model. And if the estimation of primary parameters ϕ_q', $k_{j,l}'$ is more accurate, their recursive adjustment, performed by the implementation algorithm, will be faster and more accurate as well. This means that in this case, the improved analysis works better as well.

References

1. A. Białas, R. Peschanski, Nucl. Phys. B **273**, 703 (1986); A. Białas, R. Peschanski, Nucl. Phys. B **207**, 59 (1988)
2. JACEE Coll. T.H. Burnett, et al., Phys. Rev. Lett. **50**, 2062 (1983)
3. P. Carlson (UA5), 4th Topical Workshop on P–P Collider Physics, Bern, March 1983; G.J. Alner, et al., Phys. Rep. **154**, 247 (1987)
4. For recent reviews, see E.A. de Wolf, I.M. Dremin, W. Kittel, Phys. Rep. **270**, 1 (1996); P. Bozek, M. Ploszajczak, R. Botet, Phys. Rep. **252**, 101 (1995)
5. A. Białas, R. Hwa, Phys. Lett. B **253**, 436 (1991); L. Lianshou, F. Jinghua, W. Yuanfang, HZPP **9807**
6. R. Peschanski, Nucl. Phys. B **327**, 144 (1989); P. Brax, R. Peschanski, Nucl. Phys. B **346**, 65 (1990); A. Białas, R. Peschanski, Phys. Lett. B **207**, 59 (1988); H.C. Eggers, M. Greiner, P. Lipa, Phys. Rev. Lett. **80**, 5333 (1998); H.C. Eggers, M. Greiner, P. Lipa, [hep-ph 9811204]
7. A. Białas, A. Szczerba, K. Zalewski, Z. Phys. C **46**, 163 (1990)
8. P. Desvallees, R. Ouziel, R. Peschanski, Phys. Lett. B **235**, 317 (1990); Y. Gabellini, J.-L. Meunier, R. Peschanski, Z. Phys. C **55**, 455 (1992)
9. R. Hwa, Acta Phys. Pol. B **27**, 1789 (1996)
10. A. Białas, B. Ziaja, Phys. Lett. B **378**, 319 (1996)
11. R. Janik, B. Ziaja, Acta Phys. Pol. B, 259 (1999)
12. D. Shertzer, S. Lovejoy, *Proc. IUATM Symp. on Turbulence and Chaotic Phenomena in Fluids, Kyoto, Sept. 1983* (North-Holland, Amsterdam, 1984); D. Shertzer, S. Lovejoy, *Selected Papers from the 4th Symp. on Turbulent Shear Flows*, University of Karlsruhe (1983), edited by L.J.S. Bradbury, et al. (Springer, 1984)
13. C. Meneveau, K.R. Sreenivasan, Phys. Rev. Lett. **59**, 1424 (1987)
14. R. Peschanski, J. Seixas, preprint (CERN-TH-5903-90),